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The Electron Field and the Dirac Bracket

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Abstract

A recent quantization rule of Fermi systems starts from the new symmetric brackets of classical mechanics. As a consequence, Fermi and Bose quantization can be put on an equal footing, instead of the standard *ad hoc* procedure. We prove that the rule gives the right anticommutation relations when applied to the case of the relativistic electron. We show that this is a crucial test of the rule.

For completeness, Dirac's Hamiltonian mechanics and the plus and minus Dirac bracket formalisms are developed for the electron's field.

1. *Introduction*

The symmetric brackets of classical mechanics were introduced in the last few years in order to put on equal footing the quantization of Bose and Fermi systems (Droz-Vincent, 1966; Franke & Kálnay, 1970). Let us consider systems for which Fermi-Dirac quantization is desired. If at the c-number level such systems have their phase space restricted by plus second class constraints then the plus (or symmetric) Dirac bracket is the classical bracket which goes to the anticommutator through the quantization rule (Franke & Kálnay, 1970).‡§

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‡ The reasons are similar to the ones shown by Dirac for Bose-Einstein quantization: for the last case, if at the c-number level the system has minus (or ordinary) second class constraints, which restrict its phase space, then the minus (otherwise known as antisymmetric or ordinary) Dirac bracket is the classical bracket which goes to the commutator through the quantization rule. On the other hand if a Poisson bracket is used, contradiction arises (Dirac, 1950, 1964).

§ For a short review and a comparison of the classical brackets relevant to Fermi and Bose quantization (symmetric and antisymmetric brackets for constrained or unconstrained systems) see e.g. Franke & Kálnay (1970).

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The $\pm$Dirac brackets have the form

$$\{F,G\}_{\pm}{}^{*} =^{\mathrm{df}} \{F,G\}_{\pm} - \sum_{\Gamma,\Gamma'} \{F,\theta_{\Gamma}\}_{\pm}\, c^{\pm}_{\Gamma\Gamma'} \{\theta_{\Gamma'},G\}_{\pm} \tag{1.1}$$

where

$$\{F,G\}_{\pm} =^{\mathrm{df}} \sum_{A} \left(\frac{\partial F}{\partial q_A}\frac{\partial G}{\partial p_A} \pm \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial q_A} \right) \tag{1.2}$$

are the $\pm$Poisson brackets, the $\theta_{\Gamma}(q,p)$ such that†

$$\theta_{\Gamma} \approx 0 \tag{1.3}$$

belong to a maximal set of second class constraints and $c^{\pm}$ is the matrix defined by

$$\|c^{\pm}_{\Gamma\Gamma'}\| =^{\mathrm{df}} \|\{\theta_{\Gamma},\theta_{\Gamma'}\}_{\pm}\|^{-1} \tag{1.4}$$

(Dirac, 1950, 1964; Droz-Vincent, 1966; Franke & Kálnay, 1970).

Let us have a quantum Fermi-Dirac system such that its c-number counterpart has $\pm$second class constraints (Franke & Kálnay, 1970). Then the quantization rule reads

$$\xi\{\,,\}_{+}{}^{*} \to [\,,]_{+} \tag{1.5a}$$

where ξ is an arbitrary parameter: the classical +Dirac bracket goes to the quantum anticommutator. Rule (1.1) applies to phase space coordinates. (Kálnay & Ruggeri, 1972).‡

If the system were not restricted by +second class constraints, then the quantization rule for the Fermi-Dirac case would read

$$\xi\{\,,\}_{+} \to [\,,]_{+} \tag{1.5b}$$

In a paper by Kálnay & Ruggeri (1972) it was shown that arbitrarily coupled non-relativistic oscillators can be described by a Lagrangian such that +second class constraints appear in phase space. When the quantization rule (1.5a) was used for that system, the quantum anticommutator relations of Fermi-Dirac statistics were obtained. (On the other hand, we were not able to get such Fermi quantization of oscillators without a phase constrained description of the c-number system.)§

In the present paper we test the quantization rule (1.5) *by applying it to the quantization of the relativistic electron. This is indeed a crucial test of the rule* (1.5) because for the electron field both (i) the classical starting point (the c-number Lagrangian) and (ii) the quantum relation to be obtained (the anticommutation rules of the field) by means of the quantization rule, were well known before the quantization rule (1.5) of Fermi systems was stated.

† The weak equality $\approx$ is used according to Dirac's definition (1950, 1958, 1964).

‡ In Franke & Kálnay (1970) $\xi = i$ was used, which seems possible but unnecessarily restricted. Notice that we use $\hbar = 1$.

§ Not only c-number systems exist which go to quantum Fermi systems through the quantization rule, but also c-number systems can be constructed which go to the more general set of the quantum para-Fermi case (Kálnay, 1972).

Two Lagrangian densities were widely used for the Lagrangian formalism of the relativistic electron:†

$$\mathscr{L}^{(i)} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{1.6}$$

and

$$\mathscr{L}^{(0)} = -\tfrac{1}{2}\bar{\psi}(-i\gamma^\mu \partial_\mu + m)\psi - \tfrac{1}{2}[(i\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi}]\psi \tag{1.7}$$

(see e.g. Lurié, 1968; Schweber, 1962). They differ by a 4-divergence so that, by using the Lagrangian formalism, one obtains from any of them the same set of equations of motion,

$$i\partial_\mu \gamma^\mu \psi \approx m\psi \tag{1.8a}$$

$$i\partial_\mu \bar{\psi}\gamma^\mu \approx -m\bar{\psi} \tag{1.8b}$$

i.e. Dirac's equation and its adjoint. The Lagrangians $\mathscr{L}^{(i)}$ and $\mathscr{L}^{(0)}$ are usually considered as physically equivalent.‡

For reasons to be clear later on, it will be useful to study a Lagrangian density $\mathscr{L}^{(\xi')}$ which differs from $\mathscr{L}^{(i)}$ by a more general 4-divergence $\partial_\mu V^\mu$. We impose that $\mathscr{L}^{(\xi')}$ be translation invariant and that second order derivatives be absent from the Lagrangian so that V^μ cannot contain terms like $\bar{\psi}x^\mu\psi$ and $\bar{\psi}\partial^\mu\psi$. Under these conditions the more general Lagrangian density is

$$\mathscr{L}^{(\xi')} = \tfrac{1}{2}(\xi' + i)\bar{\psi}\gamma^\mu \partial_\mu \psi + \tfrac{1}{2}(\xi' - i)(\partial_\mu \bar{\psi})\gamma^\mu \psi - m\bar{\psi}\psi \tag{1.9}$$

which is such that

$$\mathscr{L}^{(\xi')} = \mathscr{L}^{(i)} + \tfrac{1}{2}(\xi' - i)\,\partial_\mu(\bar{\psi}\gamma^\mu \psi) = \mathscr{L}^{(0)} + \tfrac{1}{2}\xi'\,\partial_\mu(\bar{\psi}\gamma^\mu \psi) \tag{1.10}$$

and

$$\mathscr{L}^{(\xi')} = (1 + i\xi')\mathscr{L}^{(0)} - \xi' i\mathscr{L}^{(i)} \tag{1.11}$$

so that $\mathscr{L}^{(\xi')}$ encompasses the Lagrangians (1.6) and (1.7). In equation (1.9) ξ' is an arbitrary complex parameter. The Lagrangian formalism leads again [as it obviously should because of equations (1.10) or (1.11)] to the equations of motion (1.8).

According to the rules of field theory for *charged* systems, we use as configuration space variables the fields $\psi_r(t,\mathbf{x})$, $\bar{\psi}_r(t,\mathbf{x})$. The canonical momenta densities conjugated to them, namely§

$$\Pi = \partial\mathscr{L}/\partial\dot{\psi}, \qquad \bar{\Pi} = \partial\mathscr{L}/\partial\dot{\bar{\psi}} \tag{1.12}$$

can be written as

$$\begin{aligned} \Pi_r(t,\mathbf{x}) &= \tfrac{1}{2}(\xi' + i)[\bar{\psi}(t,\mathbf{x})\gamma^0]_r = \tfrac{1}{2}(\xi' + i)\psi_r^{+}(t,\mathbf{x}) \\ \bar{\Pi}_r(t,\mathbf{x}) &= \tfrac{1}{2}(\xi' - i)[\gamma^0\psi(t,\mathbf{x})]_r \end{aligned} \tag{1.13}$$

† For Dirac's equation, space time metric and relativistic indices we use the conventions by Messiah (1960).

‡ A discussion of their equivalence can be done as in Kálnay & Ruggeri (1972).

§ Given any quantity A associated to ψ we denote by $\bar{A}$ the corresponding quantity associated to $\bar{\psi}$. This does *not* imply necessarily $\bar{A} = A^{+}\gamma^0$. Cf. equation (1.13).

so that (as it is well known) there are phase space constraints for the relativistic electron: the fields

$$\begin{aligned}\theta_r(t,\mathbf{x}) &= \Pi_r(t,\mathbf{x}) - \tfrac{1}{2}(\xi' + i)[\bar{\psi}(t,\mathbf{x})\gamma^0]_r \\ \bar{\theta}_r(t,\mathbf{x}) &= \bar{\Pi}_r(t,\mathbf{x}) - \tfrac{1}{2}(\xi' - i)[\gamma^0\psi(t,\mathbf{x})]_r\end{aligned} \tag{1.14a}$$

are weakly equal to zero†

$$\theta \approx 0, \qquad \bar{\theta} \approx 0 \tag{1.14b}$$

The ±Poisson brackets of two functionals of the phase space variables are

$$\{F,G\}_\pm = \int d^3x \sum_{s=1}^{4} \left[\frac{\delta F}{\delta\psi_s(t,\mathbf{x})}\frac{\delta G}{\delta\Pi_s(t,\mathbf{x})} + \frac{\delta F}{\delta\bar{\psi}_s(t,\mathbf{x})}\frac{\delta G}{\delta\bar{\Pi}_s(t,\mathbf{x})} \pm \frac{\delta F}{\delta\Pi_s(t,\mathbf{x})}\frac{\delta G}{\delta\psi_s(t,\mathbf{x})} \pm \frac{\delta F}{\delta\bar{\Pi}_s(t,\mathbf{x})}\frac{\delta G}{\delta\bar{\psi}_s(t,\mathbf{x})}\right] \tag{1.15}$$

where the functional derivatives are computed as if all phase space coordinates were independent. (This is a systematic rule for Dirac's analytic mechanics of constrained systems). We shall need

$$\{\theta_r(t,\mathbf{x}), \theta_{r'}(t,\mathbf{x}')\}_\pm = \{\bar{\theta}_r(t,\mathbf{x}), \bar{\theta}_{r'}(t,\mathbf{x}')\}_\pm = 0 \tag{1.16a}$$

$$\{\theta_r(t,\mathbf{x}), \bar{\theta}_{r'}(t,\mathbf{x}')\}_\pm = \pm\{\bar{\theta}_{r'}(t,\mathbf{x}'), \theta_r(t,\mathbf{x})\}_\pm = \lambda_\pm \gamma^0_{r'r}\,\delta(\mathbf{x}-\mathbf{x}') \tag{1.16b}$$

where we write

$$\lambda_+ = -\xi', \qquad \lambda_- = -i \tag{1.16c}$$

Though the main purpose of the present paper is to compute Dirac brackets (in order to check the quantization rule (1.5)) for completeness we shall first develop Dirac's Hamiltonian mechanics starting from the Lagrangian $\mathscr{L}^{(\xi)}$; this will be done in the next section. In Section 3 we shall compute the Dirac brackets and in the last section we shall quantize the electron's field through rule (1.5) and discuss the results.

2. *Dirac's Hamiltonian Formalism*

For the standard Hamiltonian formalism it is essential that the phase space coordinates be independent. This is not our case because of the constraints (1.14). On the other hand, Dirac's extension of the Hamiltonian formalism was invented just in order to deal with such systems (Dirac, 1950, 1958, 1964, 1966). We shall apply this formalism to the present case. The phase space variables are $\psi_r(t,\mathbf{x})$, $\bar{\psi}_r(t,\mathbf{x})$, $\Pi_r(t,\mathbf{x})$, $\bar{\Pi}_r(t,\mathbf{x})$ (charged field).

From equations (1.9) and (1.13) we obtain the standard Hamiltonian

$$H(t) = H(0) = \int d^3x\,\mathscr{H}(t,\mathbf{x}) \tag{2.1a}$$

† Notice the difference with the standard configuration space constraints $\phi(q,\dot{q}) = 0$. The phase space constraints (1.14) are constraints for the Hamiltonian (but not for the Lagrangian) formalism.

where we write

$$\mathscr{H} = m\bar{\psi}\psi - i\bar{\psi}\nabla.\gamma\psi \tag{2.1b}$$

In addition to the constraints (1.14) further constraints (called secondary ones) may appear as a consequence of $\dot{\theta} \approx \dot{\bar{\theta}} \approx 0$ which, in turn, take the form of self consistency conditions (Dirac, 1950, 1958, 1964). In the present problem they are†

$$\{\theta_r(t,\mathbf{x}), \mathscr{H}\}_- + \sum_{s=1}^{4} \int d^3y[u_s(t,\mathbf{y})\{\theta_r(t,\mathbf{x}), \theta_s(t,\mathbf{y})\}_- + \\ + \bar{u}_s(t,\mathbf{y})\{\theta_r(t,\mathbf{x}), \bar{\theta}_s(t,\mathbf{y})\}_-] \approx 0 \tag{2.2a}$$

$$\{\bar{\theta}_r(t,\mathbf{x}), \mathscr{H}\}_- + \sum_{s=1}^{4} \int d^3y[u_s(t,\mathbf{y})\{\bar{\theta}_r(t,\mathbf{x}), \theta_s(t,\mathbf{y})\}_- + \\ + \bar{u}_s(t,\mathbf{y})\{\bar{\theta}_r(t,\mathbf{x}), \bar{\theta}_s(t,\mathbf{y})\}_-] \approx 0 \tag{2.2b}$$

where u, $\bar{u}$ are *non-canonical variables* which, together with the phase space coordinates ψ, $\bar{\psi}$, Π, $\bar{\Pi}$ are the variables of Dirac's Hamiltonian formalism. Because of equations (1.14a), (1.16), (2.1) and (2.2) we deduce

$$u \approx -\mathrm{im}\,\gamma^0\psi - \boldsymbol{\alpha}.\nabla\psi \tag{2.3a}$$

$$\bar{u} \approx \mathrm{im}\,\bar{\psi}\gamma^0 + \nabla.\bar{\psi}\boldsymbol{\alpha} \tag{2.3b}$$

These equations belong to the third group mentioned by Dirac (1964, p. 14) so that no secondary constraints exist: all constraints are given by equations (1.14).

Dirac's *total Hamiltonian* becomes

$$H_T(t) = H_T(0) = \int d^3x\mathscr{H}_T(t,\mathbf{x}) \tag{2.4a}$$

where we have written

$$\mathscr{H}_T = \mathscr{H} + \sum_{r=1}^{4} (u_r\theta_r + \bar{u}_r\bar{\theta}_r) \tag{2.4b}$$

Time derivatives are computed as

$$\dot{G} \approx \{G, H_T\}_- \\ \approx \{G, H\}_- + \sum_{r=1}^{4} \int d^3x[u_r(t,\mathbf{x})\{G, \theta_r(t,\mathbf{x})\}_- + \\ + \bar{u}_r(t,\mathbf{x})\{G, \bar{\theta}_r(t,\mathbf{x})\}_-] \tag{2.5}$$

† In the present problem the indices Γ, A used in equations like (1.1) to (1.4) are sets of elementary indices: for example, in equation (1.2) the set of the $q_A(t)$ is the set of the $\psi_r(t,\mathbf{x})$, $\bar{\psi}_r(t,\mathbf{x})$ so that we can put $A = (A, r, \mathbf{x})$, $q_A \equiv \psi_A$, where $A = \mathrm{I}, \mathrm{II}$, $r = 1, 2, 3, 4$ and $\psi_{\mathrm{I}r}(t,\mathbf{x}) = \psi_r(t,\mathbf{x})$, $\psi_{\mathrm{II}r}(t,\mathbf{x}) = \bar{\psi}_r(t,\mathbf{x})$. Similarly, the set of the Γ used for instance, equation (1.1) is the set of labels of the constraints (1.14), so that we can put again

$$\Gamma = (A, r, \mathbf{x}), \theta_{\mathrm{I}r}(t,\mathbf{x}) = \theta_r(t,\mathbf{x}), \theta_{\mathrm{II}r}(t,\mathbf{x}) = \bar{\theta}_r(t,\mathbf{x}), \qquad \text{(cf. footnote § on p. 121)}$$

It is useful to take into account these remarks when doing the calculations which lead to the results of Sections 2 and 3.

In particular, by using also equations (1.14a), (1.15) and (2.1) we deduce

$$\psi \approx u, \qquad \bar{\psi} \approx \bar{u} \tag{2.6}$$

from which the physical meaning (for the present problem) of the non-canonical variables u, $\bar{u}$ can be seen. Equations (2.3) and (2.6) imply equations (1.8), i.e. Dirac's equation and its adjoint.

3. *Dirac Brackets*†

Let us exclude $\xi' = 0$ for symmetric brackets. (No restrictions are put on ξ' for antisymmetric brackets.) Then, because of equation (1.16), no constraints $\theta_r(t,\mathbf{x})$, $\bar{\theta}_r(t,\mathbf{x})$ exist which have zero $\pm$Poisson brackets with the set of all constraints. This means that all constraints (1.14) are, for the present case, plus as well as minus second class constraints. It can be easily shown that none of these $\pm$second class constraints can be eliminated from the second class set by means of linear combinations, so that the final maximal subset of second class constraints to be used in equation (1.1) equals the set of the original constraints (1.14). Cf. Dirac (1950, 1964), Franke & Kálnay (1970).

As mentioned before‡ each of the indices Γ, Γ' is a set of elementary indices. Writing them explicitly the definition (1.1) of the $\pm$Dirac brackets reads

$$\{F,G\}_\pm{}^* = \{F,G\}_\pm - \sum_{A,A'=\mathrm{I}}^{\mathrm{II}} \sum_{r,r'=1}^{4} \int d^3x \int d^3x' \times \times \{F,\theta_A(t,\mathbf{x})\}_\pm \, c^\pm_{A,r,\mathbf{x};A',r',\mathbf{x}'} \{\theta_{A'r'}(t,\mathbf{x}'),G\}_\pm \tag{3.1}$$

The matrices $c^\pm$ can be easily computed from equations (1.4) and (1.16). Replacing this result into equation (3.1) we obtain§

$$\{F,G\}_\pm{}^* = \{F,G\}_\pm \mp \lambda_\pm^{-1} \sum_{r,r'=1}^{4} \int d^3x \{F,\theta_r(t,\mathbf{x})\}_\pm \gamma^0_{rr'} \{\bar{\theta}_{r'}(t,\mathbf{x}),G\}_\pm - - \lambda_\mp^{-1} \sum_{r,r'=1}^{4} \int d^3x \{F,\bar{\theta}_r(t,\mathbf{x})_\pm \gamma^0_{r'r} \{\theta_{r'}(t,\mathbf{x}),G\}_\pm \tag{3.2}$$

Therefore, the plus Dirac brackets of phase space coordinates are, ($\xi' \neq 0$)

$$\{\psi_s(t,\mathbf{x}),\psi_{s'}(t,\mathbf{x}')\}_+{}^* = \{\bar{\psi}_s(t,\mathbf{x}),\bar{\psi}_{s'}(t,\mathbf{x}')\}_+{}^* = 0 \tag{3.3a}$$

$$\{\psi_s(t,\mathbf{x}),\bar{\psi}_{s'}(t,\mathbf{x}')\}_+{}^* = \xi'^{-1}\gamma^0_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.3b}$$

$$\{\psi_s(t,\mathbf{x}),\Pi_{s'}(t,\mathbf{x}')\}_+{}^* = [1-(1/2)\,\xi'^{-1}(\xi'-i)]\,\delta_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.4a}$$

† We only need plus Dirac brackets in order to check the quantization rule (1.5). However it will be useful, for later discussion, to compute also the minus Dirac brackets.

‡ See footnote on p. 123.

§ See footnote on p. 123.

$$\{\psi_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_+^* = \{\bar{\psi}_s(t,\mathbf{x}), \Pi_{s'}(t,\mathbf{x}')\}_+^* = 0 \tag{3.4b}$$

$$\{\bar{\psi}_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_+^* = [1 - (1/2)\,\xi'^{-1}(\xi' + i)]\,\delta_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.4c}$$

$$\{\Pi_s(t,\mathbf{x}), \Pi_{s'}(t,\mathbf{x}')\}_+^* = \{\bar{\Pi}_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_+^* = 0 \tag{3.5a}$$

and

$$\{\Pi_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_+^* = (1/4)\,\xi'^{-1}(\xi'^2 + 1)\,\gamma^0_{s's}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.5b}$$

The corresponding minus Dirac brackets are

$$\{\psi_s(t,\mathbf{x}), \psi_{s'}(t,\mathbf{x}')\}_-^* = \{\bar{\psi}_s(t,\mathbf{x}), \bar{\psi}_{s'}(t,\mathbf{x}')\}_-^* = 0 \tag{3.6a}$$

$$\{\psi_s(t,\mathbf{x}), \bar{\psi}_{s'}(t,\mathbf{x}')\}_-^* = -i\gamma^0_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.6b}$$

$$\{\psi_s(t,\mathbf{x}), \Pi_{s'}(t,\mathbf{x}')\}_-^* = [1 - (i/2)(\xi' - i)]\,\delta_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.7a}$$

$$\{\psi_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_-^* = \{\bar{\psi}_s(t,\mathbf{x}), \Pi_{s'}(t,\mathbf{x}')\}_-^* = 0 \tag{3.7b}$$

$$\{\bar{\psi}_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_-^* = [1 + (i/2)(\xi' + i)]\,\delta_{ss'}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.7c}$$

$$\{\Pi_s(t,\mathbf{x}), \Pi_{s'}(t,\mathbf{x}')\}_-^* = \{\bar{\Pi}_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_-^* = 0 \tag{3.8a}$$

and

$$\{\Pi_s(t,\mathbf{x}), \bar{\Pi}_{s'}(t,\mathbf{x}')\}_-^* = (i/4)(\xi'^2 + 1)\,\gamma^0_{s's}\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.8b}$$

One may verify that the results are consistent with the general property (Dirac, 1964; Franke & Kálnay, 1970) of the $\pm$Dirac brackets,

$$\{F, \theta_s(t,\mathbf{x})\}_\pm^* = \{F, \bar{\theta}_s(t,\mathbf{x})\}_\pm^* = 0, \ \forall F. \tag{3.9}$$

4. *Discussion*

(*i*) *When the values* (3.3) *of the* $\pm$*Dirac brackets of the c-number fields are replaced in the quantization rule* (1.5*a*), *the standard quantum anticommutation relations,*

$$[\psi_{s,op}(\mathbf{x},t), \psi_{s',op}(\mathbf{x}',t)]_+ = [\bar{\psi}_{s,op}(\mathbf{x},t), \bar{\psi}_{s',op}(\mathbf{x}',t)]_+ = 0 \tag{4.1a}$$

$$[\psi_{s,op}(\mathbf{x},t), \bar{\psi}_{s',op}(\mathbf{x}',t)]_+ = \gamma^0_{ss'}\,\delta(\mathbf{x}-\mathbf{x}')\,I_{op} \tag{4.1b}$$

of the relativistic electron are retrieved if we identify

$$\xi = \xi' \tag{4.2}$$

(*ii*) If we choose $\xi' = i$, i.e. if the starting point is the popular Lagrangian $\mathscr{L}^{(i)}$ defined in equation (1.6) then the value $\xi = i$ must be used in the quantization rule (1.5) (cf. Franke & Kálnay, 1970). However, this does not help to find the right value of the parameter ξ because we cannot see any reason to single out this Lagrangian with regard to the more general one defined in equation (1.9). It may also be that any non-zero value of ξ is allowed (cf. Kálnay & Ruggeri, 1972).

(*iii*) A zero value of the parameter ξ' cannot be used in the present formalism for *plus* Dirac brackets, because then the $\pm$Poisson brackets of the constraints are all zero [see equations (1.16)], i.e. all constraints are +first class and the +Dirac brackets do not exist (cf. Franke & Kálnay, 1970). As a result, the Lagrangian $\mathscr{L}^{(0)}$ quoted in equation (1.7) (and which is as popular as $\mathscr{L}^{(i)}$) is privileged with respect to the infinite set of the Lagrangians $\mathscr{L}^{(\xi')}$ in the sense that it is the only one for which +Dirac brackets do not exist, so that the quantization rule (1.5a) cannot be applied.

(*iv*) Results (*i*) to (*iii*), concerning Fermi quantization, are similar to the ones obtained by Kálnay & Ruggeri (1972) for a non-relativistic set of oscillators and have the same consequences, which we do not repeat here.

(*v*) The same set of non-relativistic oscillators was shown (Kálnay & Ruggeri, 1972; see also Kálnay, 1972) to be consistent also with Bose quantization through Dirac's rule (1950, 1964) for systems whose phase space is restricted by minus second class constraints. Equations (3.6) to (3.8) imply that the same happens with the relativistic electron. We point out this similarity only from a formal point of view. Physical requirements deny, as it is well known, this possibility for the relativistic case.

(*vi*) When quantizing the phase space constraints (1.14) must be transformed into

$$\theta_{r,op}(t,\mathbf{x}) = 0, \qquad \bar{\theta}_{r,op}(t,\mathbf{x}) = 0 \tag{4.3}$$

From equation (3.9) we learn that no consistency problem appears after Fermi quantization (Franke & Kálnay, 1970).† As a corollary from this remark and from equation (1.14) we see that after equations (3.3) be used for the quantization rule (1.5a) no new information is obtained from equations (3.4) and (3.5).

(*vii*) On the other hand, if one insists in quantizing through +Poisson brackets according to (1.5b), then because of equations (1.14) one deduces

$$\{\psi_r(t,\mathbf{x}), \theta_s(t,\mathbf{y})\}_\pm = \{\bar{\psi}_r(t,\mathbf{x}), \bar{\theta}_s(t,\mathbf{y})\}_\pm = \delta_{rs}\,\delta(\mathbf{x}-\mathbf{y}) \neq 0 \tag{4.4}$$

so that when quantizing contradiction with equation (4.3) arises. An identical problem would appear for the formal Bose quantization through minus Poisson brackets.

(*viii*) Let us accept the identification $\xi = \xi'$. Then it follows from equation (1.5a) and from remarks (*i*) and (*iii*) that the only value of ξ' for which the quantization rule does not lead to the right anticommutation relations for the electron (i.e. $\xi' = 0$), is the one for which, for any physical system, the quantization rule becomes meaningless: If $\xi = 0$ equations (1.5) do not sense.‡§

(*ix*) *In the introduction it was shown that the present research can be considered as a crucial test for the quantization rule* (1.5): *The result is positive*, as shown in (*i*) and (*viii*).

† A similar property is known for Bose quantization in terms of minus Dirac brackets for systems whose phase space is restricted by minus second class constraints (Dirac, 1964).

‡ Moreover it follows from equations (1.5a) and (3.3) that if the limit $\xi = \xi' \to 0$ is taken for the quantization procedure, then the right result is again retrieved for the electron.

§ A similar analysis could be done for the non-relativistic oscillators previously mentioned.

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